

NDA-FP-28
September 1996

Casimir effect in the curved background and the black hole in three dimensions

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Abstract

We consider the quantum correction to the Lagrangean by the massless free boson in the curved background in three dimensions where one of the coordinates is periodic. The correction term is given by an expansion of the metric with respect to the derivative and the first term expresses to the usual Casimir energy. As an application, we investigate the change of the geometry in three dimensional black hole due to the quantum effect and we show that the geometry becomes like that of the Reissner-Nordström solution.

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Since the finite size effects, *e.g.* the Casimir effect, in the curved space were usually calculated by fixing the background geometry [1], it is not always useful to consider the quantum back reaction to the geometry. In this paper, we consider the quantum correction to the Lagrangean by the massless free boson in the curved background in three dimensions where one of the coordinates is periodic. The correction term is given by an expansion of the metric with respect to the derivative and the first term express the usual Casimir energy. As an application, we investigate the change of the geometry in three dimensional black hole due to the quantum effect and we show that the geometry becomes like that of the Reissner-Nordström solution.

The d'Alembertian \square for the free massless boson φ is defined by

$$\square \equiv \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \quad (1)$$

and the general coordinate transformation invariant measure for the boson φ is given by

$$\Delta\varphi^2 = \int dx^3 \sqrt{g} \delta\varphi(x)^2 . \quad (2)$$

If we define a new field $\tilde{\varphi}$ by

$$\tilde{\varphi} = g^{\frac{1}{4}} \varphi , \quad (3)$$

the measure becomes trivial:

$$\Delta\tilde{\varphi}^2 = \int dx^3 \delta\tilde{\varphi}(x)^2 . \quad (4)$$

Then the d'Alembertian \square for $\tilde{\varphi}$ is given by

$$\square \equiv g^{-\frac{1}{4}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu g^{-\frac{1}{4}} . \quad (5)$$

We now evaluate $-\frac{1}{2}\text{tr} \ln \square$ by the heat kernel method as follows

$$-\frac{1}{2}\text{tr} \ln \square \equiv -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\epsilon^2}^{\infty} \frac{dt}{t} \tilde{\text{tr}} e^{t\square} . \quad (6)$$

Here $\tilde{\text{tr}}$ is defined by

$$\begin{aligned} \tilde{\text{tr}} \mathcal{O} &\equiv \int d^3x \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \\ &\times e^{-i(k_1 x^1 + k_2 x^2 + n x^3)} \mathcal{O} e^{i(k_1 x^1 + k_2 x^2 + n x^3)} , \end{aligned} \quad (7)$$

and we assume that x^3 is a coordinate with the period 2π :

$$x^3 \sim x^3 + 2\pi . \quad (8)$$

When we expand $\text{tr} \ln \square$ with respect to the derivatives, we find the following expression by replacing t with $\epsilon^2 t$;

$$\begin{aligned} \text{tr} \ln \square &= \int_1^\infty \frac{dt}{(2\pi)^2 \epsilon^2 t^2} \sum_{n=-\infty}^\infty \left\{ f^{(0)} + t\epsilon^2 \sum_{m=0}^\infty f_m^{(1)} \frac{d^m}{d\alpha^m} \right. \\ &\quad \left. + \dots + (t\epsilon^2)^l \sum_{m=0}^\infty f_m^{(l)} \frac{d^m}{d\alpha^m} + \dots \right\} e^{-\alpha t \epsilon^2 n^2} \end{aligned} \quad (9)$$

Here $f_m^{(k)}$ contains $2k$ -derivatives of the metric $g_{\mu\nu}$ and

$$\alpha = \frac{1}{g_{33}} \quad (10)$$

$$f^{(0)} = \sqrt{\frac{g}{g_{33}}} . \quad (11)$$

We now evaluate the first term $f^{(0)}$,

$$\begin{aligned} &\sqrt{\frac{g}{g_{33}}} \int_1^\infty \frac{dt}{(2\pi)^2 \epsilon^2 t^2} \sum_{n=-\infty}^\infty e^{-\frac{\epsilon^2}{g_{33}} n^2 t} \\ &= \frac{\sqrt{g}}{(2\pi)^2 \epsilon^3} \int_1^\infty \frac{dt}{t^{\frac{5}{2}}} \left(1 + 2 \sum_{n=1}^\infty e^{-\frac{\pi^2 g_{33}}{\epsilon^2 t} n^2} \right) . \end{aligned} \quad (12)$$

Here we have used the modular transformation

$$1 + 2 \sum_{n=1}^\infty e^{-a\pi n^2} = \alpha^{-\frac{1}{2}} \left(1 + 2 \sum_{n=1}^\infty e^{-\frac{\pi}{a} n^2} \right) . \quad (13)$$

The first divergent term can be absorbed into the renormalization of the cosmological constant. We rewrite the remaining term as follows,

$$\begin{aligned} &\frac{2\sqrt{g}}{(2\pi)^2 \epsilon^3} \int_1^\infty \frac{dt}{t^{\frac{5}{2}}} \sum_{n=1}^\infty e^{-\frac{\pi^2 g_{33}}{\epsilon^2 t} n^2} \\ &= \frac{2\sqrt{g}}{(2\pi)^2 \pi^3 (g_{33})^{\frac{3}{2}}} \sum_{n=1}^\infty \frac{1}{n^3} \int_0^{\frac{\pi^2 g_{33} n^2}{\epsilon^2}} ds s^{\frac{1}{2}} e^{-s} \quad \left(s \equiv \frac{\pi^2 g_{33} n^2}{\epsilon^2 t} \right) \end{aligned}$$

$$\begin{aligned}
& \xrightarrow{\epsilon \rightarrow 0} \frac{2\sqrt{g}}{(2\pi)^2 \pi^3 (g_{33})^{\frac{3}{2}}} \Gamma\left(\frac{3}{2}\right) \zeta(3) \\
& = \frac{\sqrt{g}}{(2\pi)^2 \pi^{\frac{5}{2}} (g_{33})^{\frac{3}{2}}} \zeta(3)
\end{aligned} \tag{14}$$

The second term in Eq.(9)

$$\sum_{m=0} f_m^{(1)} \frac{d^m}{d\alpha^m} \int_1^\infty \frac{dt}{(2\pi)^2 t} e^{-\alpha t \epsilon^2 n^2} \tag{15}$$

can be also evaluated by using the modular transformation (13) as follows,

$$\begin{aligned}
& \int_1^\infty \frac{dt}{(2\pi)^2 t} \sum_{n=-\infty}^\infty e^{-\alpha t \epsilon^2 n^2} \\
& = \int_1^\infty \frac{dt}{(2\pi)^2 \epsilon \alpha^{\frac{1}{2}} t^{\frac{3}{2}}} \left(1 + 2 \sum_{n=1}^\infty e^{-\frac{\pi^2}{\alpha t \epsilon^2} n^2} \right)
\end{aligned} \tag{16}$$

The first divergent term can be also absorbed into the renormalization of the gravitational constant and the remaining term can be rewritten as follows,

$$\begin{aligned}
& 2 \int_1^\infty \frac{dt}{(2\pi)^2 \epsilon \alpha^{\frac{1}{2}} t^{\frac{3}{2}}} 2 \sum_{n=1}^\infty e^{-\frac{\pi^2}{\alpha t \epsilon^2} n^2} \\
& = \frac{2}{(2\pi)^2 \pi} \sum_{n=1}^\infty \frac{1}{n} \int_0^{\frac{\pi^2 n^2}{\alpha \epsilon^2}} ds s^{-\frac{1}{2}} e^{-s} \quad \left(s \equiv \frac{\pi^2 n^2}{\alpha \epsilon^2 t} \right) \\
& \xrightarrow{\epsilon \rightarrow 0} \frac{2}{(2\pi)^2 \pi^{\frac{1}{2}}} \zeta(1)
\end{aligned} \tag{17}$$

Note that the result is α -independent. Therefore $m \neq 0$ terms in Eq.(15) do not contribute. $f_0^{(1)}$ in the remaining term has the following form,

$$\begin{aligned}
f_0^{(1)} = & \sqrt{\frac{g}{g_{33}}} \left\{ g_{ij} g_{kl} g_{mn} \left(-\frac{1}{8} g^{i\mu} \partial_\mu g^{km} g^{j\nu} \partial_\nu g^{ln} - \frac{1}{16} g^{i\mu} \partial_\mu g^{kl} g^{j\nu} \partial_\nu g^{mn} \right. \right. \\
& - \frac{1}{4} g^{i\mu} \partial_\mu g^{lm} g^{k\nu} \partial_\nu g^{jn} - \frac{1}{4} g^{i\mu} \partial_\mu g^{kl} g^{m\nu} \partial_\nu g^{jn} - \frac{1}{4} g^{i\mu} \partial_\mu g^{jm} g^{k\nu} \partial_\nu g^{ln} \Big) \\
& + g_{ij} g_{kl} \left(\frac{1}{12} g^{\mu\nu} \partial_\mu g^{ij} \partial_\nu g^{kl} + \frac{1}{6} g^{\mu i} \partial_\mu g^{\nu j} \partial_\nu g^{kl} + \frac{1}{6} g^{\mu i} g^{\nu j} \partial_\mu \partial_\nu g^{kl} \right. \\
& \left. \left. + \frac{1}{4} g^{\mu i} \partial_\nu g^{\nu j} \partial_\mu g^{kl} + \frac{1}{12} g^{\mu\nu} \partial_\mu g^{ik} \partial_\nu g^{jl} + \frac{1}{6} g^{\mu i} \partial_\mu g^{\nu k} \partial_\nu g^{lj} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{6}g^{\mu i}g^{\nu k}\partial_{\mu}\partial_{\nu}g^{jl}+\frac{1}{4}g^{\mu i}\partial_{\nu}g^{\nu k}\partial_{\mu}g^{lj}) \\
& +g_{ij}\left(-\frac{1}{4}g^{\mu\nu}\partial_{\mu}\partial_{\nu}g^{ij}-\frac{1}{4}\partial_{\mu}g^{\mu\nu}\partial_{\nu}g^{ij}-\frac{1}{2}g^{\mu i}\partial_{\mu}\partial_{\nu}g^{\nu j}-\frac{1}{4}\partial_{\mu}g^{\mu i}\partial_{\nu}g^{\nu j}\right) \\
& -\frac{1}{16}g^{\mu\nu}g_{\rho\sigma}\partial_{\nu}g^{\rho\sigma}g_{\eta\zeta}\partial_{\nu}g^{\eta\zeta}+\frac{1}{4}g^{\mu\nu}\partial_{\mu}g_{\rho\sigma}\partial_{\nu}g^{\rho\sigma}+\frac{1}{4}g^{\mu\nu}\partial_{\mu}\partial_{\nu}g^{\rho\sigma}g_{\rho\sigma} \\
& +\frac{1}{4}\partial_{\mu}g^{\mu\nu}\partial_{\nu}g^{\rho\sigma}g_{\rho\sigma}\}.
\end{aligned} \tag{18}$$

Here the roman indeces $i, j, k, \dots = 1, 2$ and the Greek ones $\mu, \nu, \rho, \dots = 1, 2, 3$. Then the finite part of $-\frac{1}{2}\text{tr} \ln \square$ has the following form

$$\begin{aligned}
& -\frac{1}{2}\text{tr} \ln \square|_{\text{finite part}} \\
& = c_0 \frac{\sqrt{g}}{(g_{33})^{\frac{3}{2}}} + c_1 f_0^{(1)} + \text{higher derivative term}
\end{aligned} \tag{19}$$

Here

$$c_0 = -\frac{\zeta(3)}{2(2\pi)^2\pi^{\frac{5}{2}}}, \quad c_1 = -\frac{\zeta(1)}{(2\pi)^2\pi^{\frac{1}{2}}}. \tag{20}$$

Note that the parameter c_0 is negative but if we consider the contribution from the free fermions² instead of the free boson, c_0 can be positive.

The first term in Eq.(19) expresses the Casimir energy in three dimensions. When the three dimensional space-time is given by the direct product of a circle (S^1) and two dimensional space (R^2), g_{33} is given by

$$g_{33} = \sqrt{L}. \tag{21}$$

Here L is the radius of S^1 . If we substitute Eq.(21) into Eq.(14), the first term in Eq.(19) is proportional to $\frac{1}{L^3}\sqrt{g}$ and the second term to $\frac{1}{L}\sqrt{g}R$ and reproduce the known results [1]. The full expression for $-\frac{1}{2}\text{tr} \ln \square$ would contain an infinite number of derivatives and would be non-local³ since $-\frac{1}{2}\text{tr} \ln \square$ should depend on the global information like periodic dimensions, *e.g.* L in Eq.(21). The expression in Eq.(19) does not appear to be covariant under the general coordinate transformation. The covariance would be, however,

² The quantum back reaction of Nambu-Jona-Lasino type fermion was investigated in Ref.[2].

³ Such a non-local effective action was obtained by using renormalization group techniques in Ref.[3].

restored in a non-local way if we could obtain the full expression since the heat kernel method used here manifestly preserves the general coordinate invariance.

We now consider the change of the geometry in the three dimensional black hole [4] due to the quantum correction obtained in (19).

The geometry of the three dimensional static (non-rotating) black hole is obtained by imposing a periodic boundary condition on the anti-de Sitter space, whose metric has the following form

$$ds^2 = l^2 \left(\frac{dz^2 + d\beta^2 - d\gamma^2}{z^2} \right). \quad (22)$$

Here $\{z, \beta, \gamma\}$ are Poincaré coordinates and $\Lambda = -l^{-2}$ is the cosmological constant. The periodic boundary condition is given by the following identification,

$$(z, \beta, \gamma) \sim e^{\frac{r_0}{l}p}(z, \beta, \gamma). \quad (23)$$

Here $p = 0, \pm 2\pi, \pm 4\pi, \dots$. If we define new coordinates $\{r, \phi, t\}$ by

$$r > r_0 \quad (24)$$

$$\begin{aligned} z &= \frac{r_0}{r} e^{-\frac{r_0}{l}\phi} \\ \beta &= \left(1 - \frac{r_0^2}{r^2}\right)^{\frac{1}{2}} e^{-\frac{r_0}{l}\phi} \cosh \frac{r_0 t}{l^2} \\ \gamma &= -\left(1 - \frac{r_0^2}{r^2}\right)^{\frac{1}{2}} e^{-\frac{r_0}{l}\phi} \sinh \frac{r_0 t}{l^2}, \end{aligned} \quad (25)$$

$$r < r_0$$

$$\begin{aligned} z &= \frac{r_0}{r} e^{-\frac{r_0}{l}\phi} \\ \beta &= -\left(\frac{r_0^2}{r^2} - 1\right)^{\frac{1}{2}} e^{-\frac{r_0}{l}\phi} \cosh \frac{r_0 t}{l^2} \\ \gamma &= \left(\frac{r_0^2}{r^2} - 1\right)^{\frac{1}{2}} e^{-\frac{r_0}{l}\phi} \sinh \frac{r_0 t}{l^2}, \end{aligned} \quad (26)$$

the metric in Eq.(22) is rewritten by,

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2$$

$$\equiv g_{\mu\nu}^{(0)} dx^\mu dx^\nu . \quad (27)$$

Here the mass M is defined by

$$M = \frac{r_0^2}{l^2} . \quad (28)$$

The periodic boundary condition (23) is given in terms of these coordinates by

$$\phi \sim \phi + 2\pi \quad (29)$$

In order to evaluate $-\frac{1}{2}\text{tr} \ln \square$, we Wick-rotate the time coordinate t :

$$t = i\tau . \quad (30)$$

If we define a new radius coordinate ρ by

$$r = r_0 \cosh \frac{\rho}{l} \quad (31)$$

the metric has the following form

$$ds^2 = \left(\frac{r_0}{l}\right)^2 \sinh^2 \frac{\rho}{l} d\tau^2 + d\rho^2 + r_0^2 \cosh^2 \frac{\rho}{l} d\phi^2 . \quad (32)$$

In order to avoid the conical singularity at $\rho = 0$, the Euclid time τ should be a peiodic coordinate with the period $\frac{2\pi l^2}{r_0^2}$, which give the black hole temperature T :

$$kT = \frac{r_0}{2\pi l^2} \quad (33)$$

(k is the Boltzman constant). By further defining new coordinates (x^1, x^2, x^3) by

$$\begin{aligned} x^1 &= \rho \cos \frac{r_0 \tau}{l^2} \\ x^2 &= \rho \sin \frac{r_0 \tau}{l^2} \end{aligned} \quad (34)$$

$$x^3 = \phi, \quad (35)$$

we find that the Wick-rotated space-time is topologically $S^1 \times R^2$ (Of course this does not mean that the Wick-rotated space-time is not the direct product

of S^1 and R^2). Here the coordinate of S^1 is given by $x^3 = \phi$ and those of R^2 by (x^1, x^2) .

In the following, we solve the effective equations of motion,

$$0 = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + 2l^{-2}) + c_0 \left(-\frac{1}{2} \frac{g_{\mu\nu}}{(g_{33})^{\frac{3}{2}}} + \frac{3}{2} \frac{\delta_\mu^3 \delta_\nu^3}{(g_{33})^{\frac{1}{2}}} \right) \quad (36)$$

which is obtained from the effective Lagrangean including the quantum correction (19) when we neglect the terms containing the derivatives. By multiplying $g^{\mu\nu}$, we find the scalar curvature R is given by

$$R = -6l^{-2} . \quad (37)$$

This tells that the derivative for the metric is $O(l^{-1})$. Since the k -th term in Eq.(19) contains $2k$ -derivatives, the contribution from the k -th term is $O(l^{-2k})$. Therefore we can naturally neglect the derivative terms which contains $f_m^{(k)}$ ($k \geq 1$) in (19) when l^{-2} (minus the cosmological constant) is small.

In three dimensions, the three gauge degrees of freedom of the general coordinate transformation can be fixed by choosing the condition,

$$g_{\mu\nu} = 0 , \quad \mu \neq \nu . \quad (38)$$

The residual symmetry is given by

$$x^\mu \rightarrow f^\mu(x^\mu) . \quad (39)$$

Here f^μ does not depend on x^ν when $\mu \neq \nu$. If we assume the rotational invariance, we can also choose a coordinate system where the metric $g_{\mu\nu}$ does not depend on ϕ :

$$g_{\mu\nu} = g_{\mu\nu}(r, t) . \quad (40)$$

The radial coordinate r is defined by

$$g_{\phi\phi} = r^2 . \quad (41)$$

If we further restrict the form of the metric as follows⁴

$$g_{tt} = -f(r) , \quad g_{rr} = \frac{1}{f(r)} , \quad g_{\phi\phi} = r^2 , \quad \text{others} = 0 , \quad (42)$$

⁴ This gives the general solution when the system has the rotational invariance.

the Ricci curvatures $R_{\mu\nu}$ are given by

$$\begin{aligned}
R_{tt} &= \frac{1}{2}f(r)f''(r) + \frac{1}{2r}f(r)f'(r) \\
R_{rr} &= -\frac{1}{2f(r)}f''(r) - \frac{1}{2rf(r)}f'(r) \\
R_{\phi\phi} &= -rf'(r) \\
\text{others} &= 0
\end{aligned} \tag{43}$$

Then the solution of Eq.(36) is given by

$$f = -M + \left(\frac{r}{l}\right)^2 - \frac{c_0}{r} \tag{44}$$

Note that the metric $g_{\mu\nu}^{(0)}$ in Eq.(27) is reproduced in the limit of $c_0 \rightarrow 0$.

The geometry given by the metric (42) and (44) is crucially depend on the sign of c_0 . Since

$$f'(r) = \frac{2r}{l^2} + \frac{c_0}{r^2} \tag{45}$$

- When $c_0 > 0$ (free fermion), $f'(r) > 0$. Therefore there is only one horizon.
- When $c_0 < 0$ (free boson), $f'(r) = 0$ if $r = \tilde{r}_0 = \left(-\frac{c_0 l^2}{2}\right)^{\frac{1}{3}}$. Since

$$f(\tilde{r}_0) = -M + M_0, \quad M_0 \equiv \frac{3}{2} \left(\frac{2c_0^2}{l^2}\right)^{\frac{1}{3}} > 0 \tag{46}$$

- When $M > M_0$, there are two horizons.
- When $M < M_0$, there are no horizons.
- When $M = M_0$, there is one horizon.

Here the horizon is defined by $f(r) = 0$. When $c_0 < 0$, the behavior of the solution is very similar to that of the Reissner-Nordström solution. Especially the solution of $M = M_0$ corresponds to the extreme black hole solution. In the extreme solution, the gravity balances with the energy of electro-magnetic force. In our case, the gravity would balance with the Casimir energy. If the black hole which has the mass M greater than M_0 could lose its mass by

the Hawking radiation, the radiation would stop when the mass M becomes equal to M_0 and the extreme black hole would remain as a remnant.

Acknowledgments

The author gratefully acknowledges A. Sugamoto and S.D. Odintsov for the discussions.

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